Exponential models

Exponential functions – a table

Choose a base, e.g. 2. Write the table of the powers of 2.

Note: this is the same table you wrote in the activity about rice on the chessboard.

$$\begin{tikzcd}
0 \arrow[r, "+1"] \arrow[d] & 1 \arrow[r, "+1"] \arrow[d] & 2 \arrow[r, "+1"] \arrow[d] & 3 \arrow[d] & \dots & \dots\ n \arrow[r, "+k"] \arrow[d] & n+k \arrow[d] \\
2^0 \arrow[r, "\cdot 2"] & 2^1 \arrow[r, "\cdot 2"] & 2^2 \arrow[r, "\cdot 2"] & 2^3 & \dots & \dots\ 2^n \arrow[r, "\cdot 2^k"] & 2^n \cdot 2^k = 2^{n+k}
\end{tikzcd}$$

The arrows mean that you can "move" in the table, according to the following rules:

- on the top row: moving one step from left to right means to ________________________ ;
- on the bottom row: ____ ;
- the vertical arrow means to exponentiate (take the base ___ and the exponent written _____________).

Note: You can also make k steps instead of 1, as in the very right arrows in the diagram . In this case

____.

Following the arrows in two different ways you find the property of exponents: $2^{n+k}=$ ______________

This reads: "2 to the power (n plus k) equals (2 to the power n) times (2 to the power k)".

We also write $\exp_2(n)$ to mean 2^n. So the property can be written as:

$$\exp_2(n+k) = \exp_2(n) \cdot \exp_2(k)$$

> **Characteristic property of the exponential function:**
>
> Given any number $a>0$ (called base), there exists a unique "regular" function $f:\mathbb{R} \to \mathbb{R}$ such that
>
> $$f(1)=a \quad \wedge \quad f(n+k)=f(n)\cdot f(k)$$
>
> It is the function $f(x)=a^x$.

In particular, if you set $k=1$ in the characteristic property, you get ______________________________________

that is, $a=\dfrac{f(n+1)}{f(n)}$.

Only exponential functions have this property, so if from an experiment you have y-values of *f* in consecutive x-values, like $f(n)$, $f(n+1)$, $f(n+2)$, $f(n+3)$ and so on, you can check whether *f* is an exponential function, by computing ____

If all these values are the same, *f* is exponential and the base is ________ . To sum up:

> **Characteristic property of the exponential function, how to use it:**
>
> If $\dfrac{f(n+1)}{f(n)}=\dfrac{f(n+2)}{f(n+1)}=\dfrac{f(n+3)}{f(n+2)}=\dots$ (and we call this number *a*), then *f* is exponential with base *a*.
>
> Remark: *f* is a function $f(x)=c\cdot a^x$ and we still do not know the value of *c* but we can compute it.

Two videos on exponential growth

In the video https://www.youtube.com/watch?v=gEwzDydciWc (on bacteria, homework) you were asked to estimate the number of bacteria after 24 hours in the case they double every 20 minutes.
Since in 24 hours they double 24·3 = 72 times, a single bacterium can give 2^{72} bacteria in one day: using the approximation $2^{10}\approx 10^3$, you get about $2^{72}=(2^{10})^7\cdot 2^2\approx(10^3)^7\cdot 4=4\cdot 10^{21}$ bacteria, which is four thousand billion billion bacteria. So the estimate given in the video is correct.

In the video https://www.youtube.com/watch?v=G75rpPOenwc (on folding paper) you were asked to complete the following table:

Folds	0	1	2	3	6	7	13	
Layers	1	2	4					
Thickness	0.1 mm	0.2 mm						439 804 km

Moral: powers of 2 become huge very quickly: as the exponent grows linearly, the power becomes big and almost unmanageable.

Real examples of exponential growth:

- spread of epidemics (biomathematics, medicine)
- compound interest (economics)
- population growth or decrease (statistics, sociology)
- radioactive decay (physics, chemistry)
- loudness of sound (physics)
- fading of light (physics)

Examples / problems

1) A population is made by 200 000 people. It **decreases by 3% each year**.
   a) Find the total population in each of the first 15 years. Use the spreadsheet on a computer.
   b) Answer the following question in natural language, not with a formula: how can you get the population of a given year, knowing the population of the preceding year?
   c) What is the population after 10 years, as a percentage of the initial population?
   d) What is, in percentage, the population loss after 8 years?
   e) Would the last two answers change if the initial population were 300 000 people?
   f) How many years are necessary to get a population of 80% (or less) of the initial population? How many years are necessary to see a 30% (or more) loss? Use the table you made for question a).
   g) Suppose now that the population decreases each year by 3% of the initial value. What is the difference? How many years are necessary to see a 30% loss?
   h) Answer in natural language, not with a formula: in the situation (g), how can you get the population of a given year, knowing the population of the preceding year? Compare with answer to b).

2) You have a **compound interest rate** on your savings (= asset = money that you spare = an amount of money) if the interest is computed every fixed amount of time (e.g. every year) on the total amount of the asset; the interest is then added to the asset, so that the asset grows and the next interest is computed on a bigger asset (old asset + interest).

   Example: You have 10 000 € and each year the bank gives you 1% interest. So:

- after one year you have 10 000 € + 1% · 10 000 € = 10 100 €
- after two years you have 10 100 € + 1% · 10 100 € = 10 201 €
- …

Suppose you have savings for a total value C. A bank offers to invest your money and give you a compound interest rate of 5% each year. A second bank gives you a compound interest rate of 1,23% every three months.

a) Which bank do you choose and why?
b) Is it true that the annual compound interest rate of the second bank is equal to $4 \cdot 1.23\,\%$?

3) A financial asset gives the right to receive 75 000 euro in 5 years, with an annual compound interest rate of 4%. What is its current value (= its value as of today)?

4) In a document[1] it is written that between 1894 and 1990 the mean height of the Italian population rose by 1,06 cm every 10 years. Would you use an exponential function to model the increment? Why?

5) In the following table you can find the total population of the US between 1790 and 1920.

Year	**Population**
1790	3 929 000
1800	5 308 000
1810	7 173 000
1820	9 638 000
1830	12 866 000
1840	17 069 000
1850	23 192 000
1860	31 443 000
1870	38 558 000
1880	50 156 000
1890	62 948 000
1900	75 995 000
1910	91 972 000
1920	105 711 000

a) In many documents[2] it is stated that between 1790 and 1860 the population growth can be assumed to be exponential. Justify the assertion, using the properties of the exponential functions.
b) Would you use an exponential function to describe the population changes after 1860?
c) Suppose that the population grew at the same rate also after 1860. In this case, what would have been the population in 1920? What would the population be today?

1 http://people.unica.it/elisabettamarini/files/2009/01/materialebiod-teoria.pdf
2 e.g. G. Israel, Modelli Matematici, ed. Muzzio 2002, page 33

6) When light enters a medium such as water or glass, its intensity decreases with depth, meaning that if you go deeper, you will see less light. In a certain lake, light intensity decreases about 72% for each meter of depth, compared to the previous depth.
   a) Explain why intensity I is an exponential function of depth d measured in meters.
   b) Use a formula to express intensity I as a function of d, using I_0 to denote the intensity of light on the surface.
   c) What is, as a percentage of I_0, the intensity of light at a depth of 4.5 meters?
   d) What is, as a percentage of I_0, the intensity loss at a depth of 6 meters?
   e) * At what depth will the intensity be 10% of the intensity on the surface?
   f) Will the intensity reduce to 0 at some depth? Explain.
   g) * In the ocean, the photic zone is the region where there is sufficient light for photosynthesis. For marine phytoplankton, the photic zone extends from the surface to the depth where the intensity is about 1% of surface light. Near Cape Cod, USA, the depth of the photic zone is about 16 meters. By what percentage does light decrease for each meter near Cape Cod?

Some results and answers

1) b) you need to multiply by… c) after 10 years the population is 73.74% of the initial population; after 8 years it decreased by 21.63% d) considering only the values in the table made in a): after 8 years the population is less than 80% of the original population; after 12 years its loss is 30% g) 10 years: here the loss is linear

2) The second option gives a higher profit because… It is false because…

3) 61 644.53 euro

4) No, it is a linear growth, because…

5) a) Consider the ratio between two consecutive numbers in the table between 1790 and 1860 that is, divide the population amount in a given year by the population amount of 10 years before. This ratio is always about 1.35… b) The ratio calculated as in a) varies in the following years c) In 1920: 31 443 000 · 1.35^6 which is about 190 millions; in 2020: more than 3 billions

6) a) Each d value is the previous value times the constant 0.72. This means that if we make a table as below we see that the in the top row going right means adding 1; in the bottom row going right means multiplying by 0.72, that is $I(n+1)=I(n)\cdot 0.72$. In other words $I(n+k)=I(n)\cdot I(k)$. Note that I is measured as a fraction of I_0.

d (depth, meters)	0	1	2	3	...	n
I (intensity, I_0)	*1 = 100%*	0.72 = 72%	0.72^2	0.72^3	...	0.72^n

b) $I(n)=0.72^n\cdot I_0$ c) about 22,8% of I_0 d) the loss is about 96.1% d) at about 7 meters (solution of $0.72^d\cdot I_0=10\%\cdot I_0$) e) according to the mathematical model, the intensity will always decrease if you increase the depth, but it will never be 0 since $0.72^d\cdot I_0=0$ does not have solutions. However, in physical terms, at some depth it will become impossible to measure the intensity of light since our instruments have a limited sensibility and resolution.